On Lattices Isomorphic to their Ideal Lattices
Nick Levine, 2012-06-17 (revised 2014-03-16)

In his paper "Lattices isomorphic to their ideal lattices" (Algebra Universalis 1971 1: 71-72), Dennis Higgs notes that the embedding a→(a] from any lattice into its lattice of ideals is an isomorphism iff every ideal is principal. He goes on to prove that there exists no other isomorphism from the lattice onto its lattice of ideals unless, again, every ideal is principal. I did not find his argument totally straightforward. Here, after some thought, is my explanation of it. My references, apart from Higgs' paper, are: George Grätzer's "General Lattice Theory" (first two dozen pages, mostly available on Google Books - unfortunately the two pages not supplied happen to be his introduction to ideals); and Halmos' "Naive Set Theory". While typing this I came across a preprint of George Bergman's 2008 paper on lattices/posets and their ideal lattices/posets; it looks relevant and appears to be more general, but I haven't read it yet.

We will need a new concept which Higgs defines, and we cite his definition verbatim thus: *let us call a subset C of L a μ-chain, where μ is an ordinal, if C is of the form* $\{c_\alpha: \alpha\leq\mu\}$ *with* $c_0=1$, $c_{\alpha+1}\prec c_\alpha$ *for all* $\alpha<\mu$, *and* $c_\lambda=\bigwedge\{c_\alpha: \alpha<\lambda\}$ *for all limit ordinals* $\lambda\leq\mu$. I find the use of foo-chains (with various ordinals playing the part of foo) rather distasteful, but there's one corner of the proof where you simply cannot escape a careful comparison of how many elements separate each of two given elements from the maximal element 1, and this is as good a way of going about it as any.

Higgs starts by noting that without loss of generality the lattice L has a smallest element (because giving it one would not change the result), that in consequence the lattice of ideals Id L is complete and therefore, given a presence of a hypothesised isomorphism f from L onto Id L, so is L.

Suppose that not every element of L is compact. Let C be any chain of non-compact elements and write $x=\bigvee C$. Either $x\in C$ or $x\notin C$. If $x\notin C$ then $x>c\ \forall\ c\in C \Rightarrow x>\bigvee_i c_i\ \forall$ finite collections $\{c_i\} \subset C$. But $x\leq\bigvee C$ and so x cannot be compact; x is therefore a non-compact upper bound for C. Alternatively $x\in C$ is a non-compact upper bound anyway. So (Zorn) we can produce a maximal non-compact element: let us call it m.

Suppose next that $\exists\ X = \{x_i \in [m,1]\}$ such that $x_1<x_2<...$, and let $x=\bigvee X\equiv\bigvee_i x_i$. Then $x>x_i\ \forall i$ and so $\nexists$ a finite subset $X_f\subseteq X$: $x\leq\bigvee X_f$. But $x\leq\bigvee X$ and so it can't be compact # (x>m must be compact). We deduce that no such X exists and hence that [m,1] satisfies the ascending chain condition (any chain in [m,1] must have a maximal element).

Let C be any maximal chain in [m,1] (note that both m and 1 belong to C) with ord C = φ. Define $C_0=\{c_0\}=\{1\}$; $\forall\alpha<\varphi$ define $c_{\alpha+1}$ = max $C\backslash C_\alpha$ and $C_{\alpha+1} = C_\alpha \cup \{c_{\alpha+1}\}$; $\forall$ limit ordinals $\lambda<\varphi$ define $c_\lambda = \bigwedge_{\alpha<\lambda}c_\alpha$ and $C_\lambda = (\bigcup_{\alpha<\lambda}C_\alpha) \cup \{c_\lambda\}$. Clearly, C_0 is a 0-chain and ord C_0 = 1. Suppose for some ordinal $\nu<\varphi$, C_ρ is a ρ-chain and ord $C_\rho = \rho+1$, $\forall\ \rho<\nu$. If ν is a successor ordinal, ν=ρ+1 say, then c_ρ = min C_ρ and c_ν = max $C\backslash C_\rho$ and so $c_\nu\prec c_\rho$. If ν is a limit ordinal then $c_\nu = \bigwedge_{\alpha<\nu} c_\alpha$. So C_ν is a ν-chain and ord $C_\nu = \nu+1$; hence by induction $\exists$ μ such that C_μ is a μ-chain and ord C_μ (= μ+1) = φ. As all of the elements of C have now been exhausted we must have C_μ=C and c_μ=m: C is a μ-chain and every element in C (other than m) is — as we have already established — compact. We will say that the set C\{m} is "compact", and that C_v is compact ∀ v<μ.

There is only one 0-chain in L (namely, C_0) and this too is compact. Suppose now that ∃ v≤μ such that, ∀ α<v, every α-chain in L is compact.

If C_α⊂L is such an α-chain then its image under the isomorphism, $f(C_\alpha)$, is an α-chain too (in Id L) and in particular $f(c_\alpha)$ is compact. Consider indeed any compact ideal A. If x∈A then x ∈ (x] ⊆ $\cup_{a\in A}$(a]; if x ∈ (a] for some a∈A then x≤a∈A, ⇒ x∈A. So A = $\cup_{a\in A}$(a] and therefore A = (A] = ($\cup_{a\in A}$(a]] = $\vee_{a\in A}$(a] ⇒ A ≤ $\vee_{a\in A}$(a] ⇒ A ≤ some finite $\vee_i$(a_i] (where {a_i} ⊆ A) = ($\vee_i a_i$] ≤ A. We deduce that all compact ideals are principal; we apply this to $f(c_\alpha)$ and write $f(c_\alpha)$ = (d_α].

Let {c_α: α≤v} form a v-chain; then so (in Id L) does {$f(c_\alpha)$} = {(d_α]: α<v}. Now, (d_0] = L= (1] ⇒ d_0 = 1. ∀ α, ($d_{\alpha+1}$]⋞(d_α] ⇒(d_α] = ($d_{\alpha+1}$] ∪ {d_α}; this cannot be the case unless $d_{\alpha+1}$⋞d_α. If λ is a limit ordinal then (d_λ] = ∧{(d_α]: α<λ} = (∧{d_α: α<λ}] ⇒ d_λ = ∧{d_α: α<λ}. So {d_α: α≤ρ} is a ρ-chain too ∀ ρ<v and hence, seeing as we hypothesised that for all such ρ every ρ-chain in L is compact, so is {d_α: α<v}.

If v is a successor ordinal, v=ρ+1 say, then c_v⋞c_ρ in L ⇒ $f(c_v)$⋞$f(c_\rho)$ in Id L ⇒ $f(c_v)$⋞(d_ρ] with d_ρ compact. Consider now any compact d∈L and any ideal B such that B⋞(d]. If x∈B then x≤∨B, ⇒ x ∈ (∨B]. So B ⊆ (∨B]. Also B ⊂ (d] and so B ⊆ (d] ∩ (∨B] = (d ∧ (∨B)] ⊆ (d]. But B⋞(d] implies that precisely one of these inclusions is sharp and the other an equality. Suppose B is not principal; then we must have (d] = (d ∧ (∨B)]⇒ d = d ∧ (∨B) ⇒ d ≤ ∨B ⇒ d ≤ $\vee_i b_i$ for some finite {b_i} ⊆ B ⇒ d ∈ the ideal B # as (d] covers B. So B must have been principal and hence so is $f(c_v)$.

Alternatively v is a limit ordinal and c_v = ∧{c_α: α<v} ⇒ $f(c_v)$ = f(∧{c_α: α<v}) = ∧{$f(c_\alpha)$}. But the c_α are compact and so $f(c_v)$ = ∧{(d_α]: α<v} = (∧{d_α}] and again $f(c_v)$ is principal. In either case let us say that $f(c_v)$ = (e].

Suppose that (e] ≤ ∨X for some X ⊆ Id L, ⇒ (e] ⊆ (∪X]. ∀x ∈ the ideal (∪X], ∃ n and ∃ x_0,...x_{n-1} ∈ ∪X such that x ≤ x_0∨...∨x_{n-1}. So in particular e ≤ e_0∨...∨e_{n-1} where for each j in [0,n-1] e_j ∈ ∪X ⇒ e_j ∈ some X_j∈∪X ⇒ e_j ∈ ($\cup_i X_i$] ⇒ e_0∨...∨e_{n-1} ∈ ($\cup_i X_i$]. But x≤e ∀x∈(e] ⇒ (e] ⊆ ($\cup_i X_i$] ⇒ (e] ≤ $\vee_i X_i$ ⇒ (e] is compact ⇒ $f(c_v)$ is compact ⇒ c_v is compact.

We have derived c_v compact, given c_α compact ∀α<v. So by induction every v-chain C_v is compact, ∀v≤μ; therefore C_μ is compact too # as c_μ=m was not compact. We are forced to conclude alternatively that every element in L is compact, hence (under the isomorphism f) so is every ideal. And we've already shown that any compact ideal is principal. So if ∃ a non-principal ideal we have to abandon the isomorphism f. ▮